LECTURE 24: PATH INDEPENDENCE, POTENTIAL FUNCTIONS, AND CONSERVATIVE FIELDS

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1. Path Independence

Definition 1. Let \( F \) be a field defined on an open region \( D \) in space. If for any two points \( A \) and \( B \) in \( D \) the work \( \int_A^B F \cdot v \, dt \) done in moving from \( A \) to \( B \) is the same over all paths from \( A \) to \( B \), then the work integral is path independent in \( D \) and the field \( F \) is conservative on \( D \).

Definition 2. If \( F \) is a field defined on \( D \) and \( F = \nabla f \) for some scalar function \( f \) on \( D \), then \( f \) is called a potential function for \( F \).

Definition 3. A region \( D \) is connected if any pair of points \( A \) and \( B \) in \( D \) can be connected by a smooth curve contained in \( D \).

Theorem 1.  
1. Let \( F = (M, N, P) \) be a vector field whose components are continuous throughout an open connected region \( D \) in space. Then there exists a differentiable function \( f \) such that

\[ F = \nabla f = (\partial_x f, \partial_y f, \partial_z f) \]

if and only if for all points \( A \) and \( B \) in \( D \) the value of \( \int_A^B F \cdot v \, dt \) is independent of the path joining \( A \) to \( B \) in \( D \).

2. If the integral is independent of the path from \( A \) to \( B \), its value is

\[ \int_A^B F \cdot v \, dt = f(B) - f(A). \]

The first part of this theorem says that most of the conservative vector fields encountered in practice are the gradients of potential functions. The second part of this theorem is kind of like a Fundamental Theorem of Calculus for line integrals. (Prove that \( F = \nabla f \) implies path independence of the integral. This can also be found on p.1078 but is definitely worth going over.)

Theorem 2. The following statements are equivalent:

1. \( \int_C F \cdot v \, dt = 0 \) around every closed loop \( C \) in \( D \).
2. The field \( F \) is conservative on \( D \).
The above results certainly imply that evaluating line integrals in conservative fields is easy if we are able to figure out when a field is conservative and then find a potential function $f$.

2. Potential Functions and Conservative Fields

**Theorem 3.** Let $\mathbf{F} = (M(x, y, z), N(x, y, z), P(x, y, z))$ be a field whose component functions have continuous first partial derivatives. Then, $\mathbf{F}$ is conservative if and only if

$$
\partial_y P = \partial_z N, \quad \partial_z M = \partial_x P, \quad \text{and} \quad \partial_x N = \partial_y M.
$$

The proof of the forward implication of this theorem is on p.1080 and uses Euler’s theorem that states that if a function has continuous partial derivatives in an open region then corresponding mixed partial derivatives are equal. The backward implication of this theorem is a consequence of Stokes’s theorem, which will be covered in a future lecture.

Once we use the above theorem to verify that $\mathbf{F}$ is conservative, we then want to find the potential function $f$. We use the fact that $\mathbf{F} = \nabla f$ to get the following equations

$$
\partial_x f = M, \quad \partial_y f = N, \quad \partial_z = P.
$$

If we first integrate $M$ with respect to $x$ (holding $y$ and $z$ fixed), then we will have $f(x, y, z) = g(x, y, z) + h(y, z)$ where $\partial_x g = M$ and $h$ is a “constant” of integration is written as a function of $y$ and $z$ since it might change if $y$ and $z$ change. We then calculate $\partial_y f = \partial_y g + \partial_y h$ and set this equal to $N$ and solve for $\partial_y h$, which we then integrate with respect to $y$ (holding $z$ fixed, no $x$ should appear in the equation for $\partial_y h$). We will then have $f(x, y, z) = g(x, y, z) + h(y, z) + k(z)$ and repeat the process for $\partial_z f$ this time setting the results equal to $P$ and solving for $dk/dz$.

I leave for the students to read about Exact Differential Forms on p.1081-1082.

3. Sample Problems

**Problem 1.** Is $\mathbf{F} = (y + z, x + z, x + y)$ conservative? If so, then find a potential function $f$ for the field $\mathbf{F}$.

**Problem 2.** Is $\mathbf{F} = (y \sin z, x \sin z, xy \cos z)$ conservative? If so, then find a potential function $f$ for the field $\mathbf{F}$.

**Problem 3.** Show that the differential form in the integral is exact. Then evaluate the integral.

$$
\int_{(0,0,0)}^{(3,3,1)} 2x \, dx - y^2 \, dy - \frac{4}{1 + z^2} \, dz
$$

Hint: $\frac{d}{dz} \tan^{-1} x = \frac{1}{1 + x^2}$.
Problem 4. Show that the differential form in the integral is exact. Then evaluate the integral.

\[ \int_{(0,2,1)}^{(1,\pi/2,2)} 2 \cos y \, dx + \left( \frac{1}{y} - 2x \sin y \right) \, dy + \frac{1}{z} \, dz \]

4. Suggested Homework

Section 14.3: 3, 5, 7, 9, 15, 17, 19, 21, 29