Phase Estimation
and
Absorption Mode Images

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The Spectrum of the Data
How Absorption Mode Spectra Are Created

The complex spectrum of the data $S(\omega_i)$ are multiplied by a phase

\[ \text{Absorption Spectra} = S(\omega_i) \times \exp \{ i\phi + i\tau \omega_i \} \]

where

- $\phi$ is an estimate of the constant phase
- $\tau$ is an estimate of the peak signal intensity
MR Image Data

Real

Imaginary
The Problem

Absolute Value Mode

Absorption Mode, Unphased
The Problem, Produce Absorption Mode Images

- Estimate and remove the linearly varying first-order phase
- Estimate and remove the constant zero-order phase
Zero and First-Order Phase Estimation

In the image domain, the model equation for the zero-order phase

\[ d_i = A_i \exp\{-i\theta\} + n_i \]

and for the first-order phase

\[ d_i = A_i \exp\{-i(\theta + \tau\omega_i)\} + n_i \]

Applying the rules of probability theory, one must compute

\[ P(\theta|D\sigma I) \propto \int P(A|I)P(D|\theta A_1 \ldots A_N\sigma I)dA \]
The Prior Probability For The Amplitudes

\[ P(A|I) \propto \exp \left\{ -\frac{\beta^2}{2\sigma^2} \sum_{j=1}^{N} \sum_{k=1}^{N} A_j H_{jk} A_k \right\} \]

where \( H_{jk} \) is arbitrary.
Zero and First-Order Phase Estimation

The sufficient statistic \( h^2(\theta) \) can be written

\[
\bar{h}^2(\theta) = \frac{U + \sqrt{W^2 + X^2 \cos(2\theta + \psi)}}{2N}
\]

where

\[
U \equiv \sum_{i=1}^{N} \hat{a}_i d_{R_i} + \hat{b}_i d_{I_i} \quad W \equiv \sum_{i=1}^{N} \hat{a}_i d_{R_i} - \hat{b}_i d_{I_i}
\]

and

\[
X \equiv \sum_{i=1}^{N} \hat{a}_i d_{I_i} + \hat{b}_i d_{R_i} \quad \psi \equiv \tan^{-1} \left( \frac{X}{W} \right)
\]
Assuming $\sigma$ is known

$$P(\theta|D\sigma I) \propto \exp \left\{ \frac{\sqrt{W^2 + X^2 \cos(2\theta + \psi)}}{4\sigma^2} \right\}$$

Consequently,

$$\theta_{\text{est}} = -\frac{\psi}{2} \pm \sqrt{\frac{8\sigma^2}{\sqrt{W^2 + X^2}}}$$
Logarithm of the Posterior Probability for $\tau$

$\log P(\tau_x|DI)$

$\log P(\tau_y|DI)$
Posterior Probability for $\tau$

$P(\tau_x|DI)$

$P(\tau_y|DI)$

$(\tau_x)_{est} = 66.0215 \pm 0.0025$

$(\tau_y)_{est} = 63.9843 \pm 0.001$
The Posterior Probability for $\theta$

\[
\log P(\theta|D\sigma I)
\]

\[
P(\theta|D\sigma I)
\]
The Fourier Transform, Real and Imaginary

real, no phases applied
imaginary
Removing the First-Order $\tau_x$ Phase

real, with $\tau_x$

imaginary
Removing the First-Order $\tau_y$ Phase

real, with $\tau_x$ and $\tau_y$ \hspace{1cm} \text{imaginary}
Removing the Constant Phase $\theta$

real, with $\tau_x$, $\tau_y$ and $\theta$ imaginary
What About Gradient-Echo Images

real, with no phases

imaginary
What About Gradient-Echo Images

real, with $\tau_x$
What About Gradient-Echo Images

real, with $\tau_x$ and $\tau_y$

imaginary
What About Gradient-Echo Images
Generating Absorption Mode Images When the Phase Varies Nonlinearly

Suppose you have a pixel

\[
\text{Complex UnPhased Pixel} = R + iI
\]

You estimate the phase to be

\[
\hat{\theta} = \tan^{-1}(I/R)
\]

You remove the effects of the phase

\[
\text{Complex Phased Pixel} = (R + iI) \times \exp\{-i\hat{\theta}\}.
\]

And you're done?
Generating Absorption Mode Images

- Real = $R \cos \hat{\theta} + I \sin \hat{\theta}$ and Imag = $-R \sin \hat{\theta} + I \cos \hat{\theta}$

- $\cos \hat{\theta} = \frac{R}{\sqrt{R^2 + I^2}}$ and $\sin \hat{\theta} = \frac{I}{\sqrt{R^2 + I^2}}$

- Real Part = $\frac{R^2 + I^2}{\sqrt{R^2 + I^2}} = \sqrt{R^2 + I^2}$

- Imaginary Part = $\frac{-RI + IR}{\sqrt{R^2 + I^2}} = 0$
Gradient-Echo Images, linearly phased

real, with $\tau_x$ and $\tau_y$  imaginary
The Gaussian Approximation

\[ P(\theta|D\sigma I) \text{ SNR} = 30:1 \]

\[ P(\theta|D\sigma I) \text{ SNR} = 0:1 \]
Absorption Mode Images with Nonlinear Phases

1. Decide if a signal is present
   - Present, sample the posterior and phase positive
   - Not Present, sample the posterior and phase

2. Repeat the previous step for each pixel in the image
What About Gradient-Echo Images

real, Nonlinearly phased

imaginary

4K pixels, $0.002 \pm 0.037$

16K pixels, $5 \times 10^{-5} \pm 0.038$
Phase Deviations From Linear, i.e., The Inhomogeneous Magnetic Field

Wrapped phase

UnWrapped
Summary and Conclusions

• Absorption mode images have significant advantages

• Three phase parameters are needed to phase most spin-echo images

• When nonlinear phases are present a sample from the joint posterior probability for all of the phases in the image is used to generate an absorption mode image.

• The map of the phase can be unwrapped and is an image of the magnetic field inhomogeneities